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Chain Tensor Products and Interval Effect Algebras

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Weak and strong n-doublings ($n \in \mathrm{N}$) are defined for an effect algebra P and the concept of a normal interval algebra is introduced. It is shown that the following statements are equivalent: (1) There is a morphism from P into an interval algebra. (2) P admits a tensor product with every finite chain. (3) P has a weak n-doubling for every $n \in \mathrm{N}$. Moreover, the following are equivalent: (4) P is a normal interval algebra. (5) P admits a strong tensor product with every chain of length 2^n, $n \in \mathrm{N}$. (6) P has a strong n-doubling for every $n \in \mathrm{N}$. Finally, it is shown that if P possesses an order-determining set of states, then P is a normal interval algebra.

1. INTRODUCTION

Effect algebras (or D-posets) have recently been introduced as an algebraic structure for investigating the foundations of quantum mechanics (Cattaneo and Nisticò, 1985; Dvurečenskij and Pulmannová, 1994; Foulis and Bennett, 1994; Giuntini and Greuling, 1989; Greechie and Foulis, 1995; Kôpka, 1992; Kôpka and Chovanec, 1994; Navara and Pták, n.d.). This framework gives a unification of the operational (Davies, 1976; Davies and Lewis, 1970; Holevo, 1982; Kraus, 1983; Ludwig, 1983/1985) and quantum logic (Beltrametti and Cassinelli, 1981; Mackey, 1963; Varadarajan, 1968/1970) approaches to quantum mechanics and yields a natural definition of a tensor product, a concept that is necessary for the study of combined physical systems (Davies, 1976; Dvurečenskij, 1995; Dvurečenskij and Pulmannová, 1994; Foulis, 1989). Interval effect algebras (or interval algebras for short) form an important class of effect algebras (Bennett and Foulis, n.d.-a,b; Foulis *et al.*, 1994). These are effect algebras that are constructed from an initial interval in the positive cone of a partially ordered Abelian group. Besides

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including most of the common effect algebra examples, interval algebras encompass a powerful group-theoretic structure.

In this paper, we introduce a class of interval algebras that we call normal interval algebras. This class is still broad enough to include most of the common examples. In particular, the physically important Hilbert space effect algebras are in this class. We show that there are two properties that characterize normal interval algebras. The first is that they admit a strong tensor product with any chain of length 2^n, $n \in \mathbb{N}$. The second is that they possess doubling units that enable them to be iteratively doubled in size. We also weaken these properties to characterize effect algebras that admit a morphism into an interval algebra.

A morphism from an effect algebra into the real unit interval is called a state or probability measure (Bennett and Foulis, n.d.-a; Dvurečenskij and Pulmannová, 1994; Foulis and Bennett, 1994; Kraus, 1983; Navara and Pták, n.d.). States provide a mechanism for describing the statistical properties of a quantum system. We shall show that an effect algebra possesses a state if and only if it admits a tensor product with every finite chain. Moreover, we shall give a sufficient condition that an effect algebra is a normal interval algebra in terms of the richness of its state space.

2. DEFINITIONS

This section collects some basic effect algebra definitions and results (Bennett and Foulis, n.d.-a; Dvurečenskij, 1995; Foulis and Bennett, 1994; Greechie and Foulis, 1995). An *effect algebra* $(P, \oplus, 0, 1)$ (or simply P) is a set P together with two distinct elements $0, 1 \in P$ and a partial binary operation $\oplus: D \to P$ with domain $D \subseteq P \times P$ such that: (i) $(a, b) \in D$ implies $(b, a) \in D$ and $b \oplus a = a \oplus b$; (ii) $(b, c) \in D$ and $(a, b \oplus c) \in D$ imply $(a, b) \in D$, $(a \oplus b, c) \in D$, and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$; (iii) $a \in P$ implies there exists a unique $a' \in P$ such that $(a, a') \in D$ and $a \oplus a' = 1$; (iv) $(a, 1) \in D$ implies $a = 0$.

If $(a, b) \in D$, we write $a \perp b$ and we write $a \leq b$ if there exists a $c \in P$ such that $a \oplus c = b$. It can be shown that $a \perp b$ if and only if $a \leq b'$. Moreover, $(P, \leq, 0, 1, ')$ is a bounded poset such that $a \leq b$ implies that $b' \leq a'$ and $a'' = a$ for all $a \in P$. It follows from (ii) that we can write $b = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ without parentheses whenever it is defined. In this case, if $a_i = a$, $i = 1, \ldots, n$, and b is defined, we write $b = na$.

For effect algebras P, Q, a map $\phi: P \to Q$ is said to be (i) *additive* if $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$; (ii) a *morphism* if ϕ is additive and $\phi(1) = 1$; (iii) a *monomorphism* if ϕ is a morphism and $\phi(a) \perp \phi(b)$ implies $a \perp b$; (iv) an *isomorphism* if ϕ is a surjective monomorphism.

Let P, Q, and R be effect algebras. A map $\beta: P \times Q \to R$ is a *bimorphism* if for every $a \in P$, $b \in Q$, $\beta(a, \cdot)$ and $\beta(\cdot, b)$ are additive and $\beta(1, 1) = 1$. A bimorphism $\beta: P \times Q \to R$ is *strong on* $P(Q)$ if the morphism $\beta(\cdot, 1)$ $(\beta(1, \cdot))$ is a monomorphism. If β is strong on P and Q, we say that β is *strong*.

A simple example of an effect algebra is $[0, 1] \subseteq \mathbb{R}$, where $a \perp b$ if and only if $a + b \leq 1$, in which case $a \oplus b = a + b$. Another example is an *n-chain*

$$C_n = \{0, a, 2a, \ldots, na = 1\}$$

It is clear that any two n-chains are isomorphic, so we can assume that C_n is a sub-effect algebra of $[0, 1]$ and that $a = n^{-1}$. A morphism $\phi: P \to [0, 1]$ is called a *state* and we denote the set of states on P by $\Omega(P)$. If $\Omega(P) \neq \varnothing$, we call P *stately* and if $\Omega(P) = \varnothing$, we call P *stateless*. There are examples of stateless effect algebras (Greechie, 1971; Gudder and Greechie, n.d.). Also, $[0, 1]$ and C_n are stately and their unique state is the identity function. A set of states S on P is *order determining* if $s(a) \leq s(b)$ for all $s \in S$ implies $a \leq b$.

Let P, Q, and T be effect algebras and let $\tau: P \times Q \to T$ be a bimorphism. We call (T, τ) a *tensor product* of P and Q if (i) for any bimorphism $\beta: P \times Q \to R$, there exists a morphism $\phi: T \to R$ such that $\beta = \phi \circ \tau$; (ii) every element of T is a finite sum of elements of the form $\tau(a, b)$. The tensor product is unique to within an isomorphism if it exists. We then write $T = P \otimes Q$, $\tau(a, b) = a \otimes b$ and say that $P \otimes Q$ exists. It can be shown that if P and Q admit a bimorphism or if P and Q are stately, then $P \otimes Q$ exists (Dvurečenskij, 1995; Dvurečenskij and Pulmannová, 1994). However, there are effect algebras whose tensor product does not exist (Gudder and Greechie, n.d.). It is easy to show that $C_m \otimes C_n = C_{mn}$, where $im^{-1} \otimes jn^{-1} = ij(mn)^{-1}$, $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n$. If $P \otimes Q$ exists and τ is strong, we say that $P \otimes Q$ is *strong*.

Lemma 2.1. (i) If P and Q admit a strong bimorphism, then $P \otimes Q$ is strong. (ii) If $\Omega(P)$ and $\Omega(Q)$ are order determining, then $P \otimes Q$ is strong. (iii) If $\Omega(P \otimes Q)$ is order determining and $P \otimes Q$ is strong, then $\Omega(P)$ and $\Omega(Q)$ are order determining. (iv) If $P \otimes Q$ exists and Q_0 is isomorphic to a sub-effect algebra of Q, then $P \otimes Q_0$ exists. Moreover, if $P \otimes Q$ is strong on P, then so is $P \otimes Q_0$.

Proof. (i) Let $\beta: P \times Q \to R$ be a strong bimorphism. Since a bimorphism exists, $P \otimes Q$ exists. By the definition of a tensor product there exists a morphism $\phi: P \otimes Q \to R$ such that $\beta(a, c) = \phi(a \otimes c)$ for all $a \in P$, $c \in Q$. Suppose that $a \otimes 1 \perp b \otimes 1$. Then, since $\phi(a \otimes 1) \perp \phi(b \otimes 1)$, we have $\beta(a, 1) \perp \beta(b, 1)$. Since β is strong, we conclude that $a \perp b$. Similarly, if $1 \otimes c \perp 1 \otimes d$, we have $c \perp d$. Hence, $P \otimes Q$ is strong. (ii) Suppose

$\Omega(P)$ and $\Omega(Q)$ are order determining. For $s \in \Omega(P)$, $t \in \Omega(Q)$, define the bimorphism $\beta_{s,t}$: $P \times Q \to [0, 1]$ by $\beta_{s,t}(a, b) = s(a)t(b)$. By definition of the tensor product there exists a morphism $\phi_{s,t}$: $P \otimes Q \to [0, 1]$ such that

$$s(a)t(c) = \beta_{s,t}(a, c) = \phi_{s,t}(a \otimes c)$$

for all $a \in P$, $c \in Q$. If $a \otimes 1 \perp b \otimes 1$, then $\phi_{s,t}(a \otimes 1) \perp \phi_{s,t}(b \otimes 1)$ for every $s \in \Omega(P)$, $t \in \Omega(Q)$. Hence, $s(a) + s(b) \leq 1$, so $s(a) \leq s(b')$ for every $s \in \Omega(P)$. Since $\Omega(P)$ is order determining, we conclude that $a \perp b$. Similarly, if $1 \otimes c \perp 1 \otimes d$, then $c \perp d$. Hence, $P \otimes Q$ is strong. (iii) Suppose $\Omega(P \otimes Q)$ is order determining and $P \otimes Q$ is strong. For $s \in \Omega(P \otimes Q)$, define the state $\mu_s \in \Omega(P)$ by $\mu_s(a) = s(a \otimes 1)$. Suppose $\mu_s(a) \leq \mu_s(b)$ for every $s \in \Omega(P \otimes Q)$. Then $s(a \otimes 1) \leq s(b \otimes 1)$ for every $s \in \Omega(P \otimes Q)$ and since $\Omega(P \otimes Q)$ is order determining, $a \otimes 1 \leq b \otimes 1$. Hence,

$$a \otimes 1 \leq b'' \otimes 1 = (b' \otimes 1)'$$

so $a \otimes 1 \perp b' \otimes 1$. Since $P \otimes Q$ is strong, we have $a \perp b'$, so $a \leq b$. We conclude that $\Omega(P)$ is order determining. Similarly, $\Omega(Q)$ is order determining. (iv) Let ϕ: $Q_0 \to Q$ be an isomorphism to a sub-effect algebra of Q. Define β: $P \times Q_0 \to P \otimes Q$ by $\beta(a, b) = a \otimes \phi(b)$. Then clearly β is a bimorphism, so $P \otimes Q_0$ exists. If $\beta(a, 1) \perp \beta(b, 1)$, then $a \otimes 1 \perp b \otimes 1$. Assuming that $P \otimes Q$ is strong on P, we have $a \perp b$. Hence, β is strong on P and it follows from Part (i) that $P \otimes Q_0$ is strong on P. ■

Let G be an additively written, partially ordered Abelian group (Bennett and Foulis, n.d.-a; Fuchs, 1963; Goodearl, 1986). Let $u \in G$ with $u > 0$ and let

$$P = G^+[0, u] = \{g \in G: 0 \leq g \leq u\}$$

Then P can be organized into an effect algebra $(P, \oplus, 0, u)$ by defining $a \oplus b$ if and only if $a + b \leq u$, in which case $a \oplus b = a + b$. In the effect algebra P we have $a' = u - a$ and the effect algebra partial order on P coincides with the restriction to P of the partial order on G. An effect algebra of the form $G^+[0, u]$ (or isomorphic to an effect algebra of this form) is called an *interval effect algebra* or, for short, an *interval algebra*. Notice that $[0, 1] = \mathbb{R}^+[0, 1]$ and $C_n = Z^+[0, n]$ are interval algebras. Since an interval algebra is stately (Bennett and Foulis, n.d.-a), an effect algebra admits a morphism into an interval algebra if and only if it is stately.

If $(P, \hat{\oplus}, 0, 1)$ is an effect algebra and $0 \neq u \in P$, let

$$P[0, u] = \{a \in P: 0 \leq a \leq u\}$$

Then $(P[0, u], \oplus, 0, u)$ is an effect algebra, where $a \oplus b$ is defined if and only if $a \hat{\oplus} b \leq u$, in which case $a \oplus b = a \hat{\oplus} b$. Finally, if P_i, $i = 1, \ldots, n$, are effect algebras, it is easy to show that their Cartesian product $P_1 \times$

$\cdots \times P_n$ is an effect algebra with its componentwise partial operation (Bennett and Foulis, n.d.-a; Foulis *et al.*, 1994).

3. TENSOR CHAIN ALGEBRAS

An effect algebra P is a *tensor chain algebra* if $P \otimes C_n$ exists for every $n \in \mathrm{N}$.

Lemma 3.1. An effect algebra P is a tensor chain algebra if and only if $P \otimes C_n$ exists for every $n \in I$, where $I \subseteq \mathrm{N}$ is infinite.

Proof. Suppose that $P \otimes C_n$ exists for every $n \in I$, where $I \subseteq \mathrm{N}$ is infinite. Let $m \in \mathrm{N}$ and let $n \in I$ with $m \leq n$. Defining

$$\beta\colon \quad P \times C_m \to P \otimes C_n[0, 1 \otimes mn^{-1}]$$

by $\beta(a, jm^{-1}) = a \otimes jn^{-1}$, we see that β is a bimorphism. Hence, $P \otimes C_m$ exists. ■

Suppose that $P \otimes C_n$ exists for every $n \in I$, where $I \subseteq \mathrm{N}$ is infinite. We say that

$$a_1 \otimes n^{-1} \oplus \cdots \oplus a_n \otimes n^{-1} \in P \otimes C_n$$

where $n \in I$ is *I-irreducible* if $a_1 \otimes m^{-1} \oplus \cdots \oplus a_n \otimes m^{-1}$ is not defined for $m \in I$ with $m < n$.

Lemma 3.2. If $P \otimes C_n$ exists for every $n \in I$, where $I \subseteq \mathrm{N}$ is infinite, then for every $a_i \in P$, $i = 1, \ldots, j$, there exists a unique $m \in I$ such that $a_1 \otimes m^{-1} \oplus \cdots \oplus a_j \otimes m^{-1}$ is I-irreducible.

Proof. Assume the hypothesis of the lemma and let $a_i \in P$, $i = 1, \ldots, j$. If $n \in I$ with $j \leq n$, then $a_1 \otimes n^{-1} \oplus \cdots \oplus a_j \otimes n^{-1}$ is defined in $P \otimes C_n$. Indeed, applying the effect algebra axioms, we have

$$1 \otimes jn^{-1} = 1 \otimes n^{-1} \oplus \cdots \otimes 1 \oplus n^{-1} \qquad (j \text{ summands})$$

$$= [a_1 \otimes n^{-1} \oplus a_1' \otimes n^{-1}] \oplus \cdots \oplus [a_j \otimes n^{-1} \oplus a_j' \otimes n^{-1}]$$

Since $a_1 \otimes n^{-1} \oplus \cdots \oplus a_j \otimes n^{-1}$ is a subsum of the right side, this sum is defined (Dvurečenskij, 1995; Foulis and Bennett, 1994). Letting m be the smallest integer in I such that $a_1 \otimes m^{-1} \oplus \cdots \oplus a_j \otimes m^{-1}$ is defined, we have the result. ■

Theorem 3.3. An effect algebra P admits a morphism into an interval algebra if and only if P is a tensor chain algebra.

Proof. If P admits a morphism into an interval algebra, then P is stately. Hence, P is a tensor chain algebra. Conversely, suppose P is a tensor chain algebra. Let $A_j = P \otimes C_{2^j}$, $j \in J = \{0, 1, 2, \ldots\}$ and notice that $A_0 = P$. Let $I = \{2^j: j \in J\}$ and define

$$F = \{a \in \cup A_j: a \text{ is } I\text{-irreducible}\}$$

For $a = a_1 \otimes 2^{-m} \oplus \cdots \oplus a_j \otimes 2^{-m}$, $b = b_1 \otimes 2^{-n} \oplus \cdots \oplus b_k \otimes 2^{-n} \in F$, we define $a + b = c$, where

$$c = a_1 \otimes 2^{-r} \oplus \cdots \oplus a_j \otimes 2^{-r} \oplus b_1 \otimes 2^{-r} \oplus \cdots \oplus b_k \otimes 2^{-r}$$

is I-irreducible. Then $P \subseteq F$, $+$ is a commutative binary operation on F and $0 \in P$ is an additive identity. Moreover, it is easy to show that $+$ is associative, so $(F, +, 0)$ is a commutative monoid. If $a + b = 0$, then $a, b \in P$ and $a \oplus b = 0$. Since P is an effect algebra, it follows that $a = b = 0$.

For $a, b \in F$ as given previously, we write $a \sim b$ if

$$a_1 \otimes 2^{-r} \oplus \cdots \oplus a_j \otimes 2^{-r} = b_1 \otimes 2^{-r} \oplus \cdots \oplus b_k \otimes 2^{-r} \tag{1}$$

for some $r \in J$. It is clear that $\sim$ is reflexive and symmetric. Now, if (1) holds, then it follows from properties of the tensor product that (1) holds with r replaced by p, where $r \leq p$. This observation makes it clear that $\sim$ is transitive, so $\sim$ is an equivalence relation. Moreover, this observation enables us to show that $a \sim c$, $b \sim d$ implies that $a + b \sim c + d$, so $\sim$ is a congruence relation. We denote the equivalence class containing $a \in F$ by $[a]$ and we let $\hat{F} = \{[a]: a \in F\}$. Since $\sim$ is a congruence relation, we have a well-defined operation $+$ on $\hat{F}$ given by $[a] + [b] = [a + b]$. It also follows that $\{\hat{F}, +, [0]\}$ is a commutative monoid. If $[a] + [b] = [0]$, then $[a + b] = [0] = \{0\}$. Hence, $a + b = 0$, so $a = b = 0$ and $[a] = [b] = [0]$. Thus, $\{\hat{F}, +, [0]\}$ is a positive, commutative monoid.

Now suppose that $[a] + [b] = [a] + [c]$, so $[a + b] = [a + c]$. We conclude that there exists an $r \in J$ such that

$$\begin{aligned} &a_1 \otimes 2^{-r} \oplus \cdots \oplus a_j \otimes 2^{-r} \oplus b_1 \otimes 2^{-r} \oplus \cdots \oplus b_k \otimes 2^{-r} \\ &= a_1 \otimes 2^{-r} \oplus \cdots \oplus a_j \otimes 2^{-r} \oplus c_1 \otimes 2^{-r} \oplus \cdots \oplus c_i \otimes 2^{-r} \end{aligned}$$

Applying the cancellation law for effect algebras (Foulis and Bennett, 1994), we conclude that

$$b_1 \otimes 2^{-r} \oplus \cdots \oplus b_k \otimes 2^{-r} = c_1 \otimes 2^{-r} \oplus \cdots \oplus c_i \otimes 2^{-r}$$

Hence, $b \sim c$, so $[b] = [c]$. Hence, $\{\hat{F}, +, [0]\}$ is a positive, cancellative, commutative monoid. It follows from Birkhoff's theorem (Birkhoff, 1942; Fuchs, 1963) that $\{\hat{F}, +, [0]\}$ can be enlarged to a partially ordered Abelian group $(G, +, 0)$ that has $\hat{F}$ as its positive cone. Corresponding to $1 \in P$, let $u = [1]$. For $a \in P$, since $[a] + [a'] = [1]$, we have

$$0 = [0] \leq [a] \leq [1] = u$$

Hence, $[a] \in G^+[0, u]$ and we define $\phi: P \to G^+[0, u]$ by $\phi(a) = [a]$. Then $\phi(1) = [1] = u$ and if $a, b \in P$ with $a \perp b$, we have

$$\phi(a \oplus b) = [a \oplus b] = [a + b] = [a] + [b] = \phi(a) \oplus \phi(b)$$

Therefore, ϕ is a morphism from P into the interval algebra $G^+[0, u]$. ■

We conclude from Lemma 3.1 and Theorem 3.3 that if P is stateless, then $P \otimes C_n$ does not exist for all but finitely many $n \in \mathbb{N}$. This generalizes some results in Gudder and Greechie (n.d.) and shows that there are many examples of pairs of effect algebras that do not admit a tensor product.

Lemma 3.4. For effect algebras P_i, $i = 1, \ldots, n$, their Cartesian product $P_1 \times \cdots \times P_n$ is an interval algebra if and only if each P_i is an interval algebra, $i = 1, \ldots, n$.

Proof. It is well known that the Cartesian product of a finite number of interval algebras is an interval algebra (Bennett and Foulis, n.d.-a; Foulis *et al.*, 1994). Conversely, suppose that $P_1 \times \cdots \times P_n$ is an interval algebra. Then there exists an isomorphism $\phi: P_1 \times \cdots \times P_n \to G^+[0, u]$ for some partially ordered Abelian group G. Letting $\nu = \phi(1, 0, \ldots, 0)$, we have

$$\phi(0, \ldots, 0) = 0 < \nu < u = \phi(1, \ldots, 1)$$

For $a \in P_1$, we have

$$0 \leq \phi(a, 0, \ldots, 0) \leq \phi(1, 0, \ldots, 0) = \nu$$

Define the mapping $\psi: P_1 \to G^+[0, \nu]$ by $\psi(a) = \phi(a, 0, \ldots, 0)$. Then clearly, ψ is a monomorphism. If $g \in G^+[0, \nu]$, then $g = \phi(a_1, \ldots, a_n)$ for some $a_i \in P_i$, $i = 1, \ldots, n$. Since

$$\phi(a_1, \ldots, a_n) \leq \nu = \phi(1, 0, \ldots, 0)$$

we have $(a_1, \ldots, a_n) \leq (1, 0, \ldots, 0)$, so $a_i = 0$, $i = 2, \ldots, n$. Hence, $g = \phi(a_1, 0, \ldots, 0) = \psi(a_1)$, so ψ is surjective. Therefore, ψ is an isomorphism. It follows that P_1 is an interval algebra and in a similar way, $P_2, \ldots, P_n$ are also interval algebras. ■

Lemma 3.5. For effect algebras $P_1, \ldots, P_n$, we have $s \in \Omega(P_1 \times \cdots \times P_n)$ if and only if s has the form

$$s(a_1, \ldots, a_n) = \sum_{i=1}^{n} \lambda_{j(i)}\mu_{j(i)}(a_{j(i)})$$

where $\mu_{j(i)} \in \Omega(P_{j(i)})$, $\lambda_{j(i)} > 0$, $i = 1, \ldots, m$, and $\Sigma_{i=1}^{m} \lambda_{j(i)} = 1$.

Proof. Suppose $s: P_1 \times \cdots \times P_n \to [0, 1]$ has the above form. Then

$$s(1) = s(1, \ldots, 1) = \sum \lambda_{j(i)}\mu_{j(i)}(1) = \sum \lambda_{j(i)} = 1$$

Moreover, if $(a_1, \ldots, a_n) \perp (b_1, \ldots, b_n)$, then $a_i \perp b_i$, $i = 1, \ldots, n$, and we have

$$\begin{aligned} s((a_1, \ldots, a_n) \oplus (b_1, \ldots, b_n)) &= s(a_1 \oplus b_1, \ldots, a_n \oplus b_n) \\ &= \sum \lambda_{j(i)}\mu_{j(i)}(a_{j(i)} \oplus b_{j(i)}) \\ &= \sum \lambda_{j(i)}\mu_{j(i)}(a_{j(i)}) + \sum \lambda_{j(i)}\mu_{j(i)}(b_{j(i)}) \\ &= s(a_1, \ldots, a_n) + s(b_1, \ldots, b_n) \end{aligned}$$

Hence, $s \in \Omega(P_1 \times \cdots \times P_n)$. Conversely, suppose $s \in \Omega(P_1 \times \cdots \times P_n)$. Let $\lambda_i = s(0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 appears in the ith component and let $j(i)$, $i = 1, \ldots, m$, be the indices such that $\lambda_{j(i)} \neq 0$. Define $\mu_{j(i)}$: $P_{j(i)} \to [0, 1]$ by

$$\mu_{j(i)}(a) = \lambda_{j(i)}^{-1} s(0, \ldots, 0, a, 0, \ldots, 0)$$

where a appears in the $j(i)$th component. Then $\mu_{j(i)}(1) = 1$ and if $a, b \in P_{j(i)}$ with $a \perp b$, we have

$$\begin{aligned} \mu_{j(i)}(a \oplus b) &= \lambda_{j(i)}^{-1} s(0, \ldots, 0, a \oplus b, 0, \ldots, 0) \\ &= \lambda_{j(i)}^{-1}[s((0, \ldots, 0, a, 0, \ldots, 0) \oplus (0, \ldots, 0, b, 0, \ldots, 0))] \\ &= \lambda_{j(i)}^{-1} s(0, \ldots, 0, a, 0, \ldots, 0) + \lambda_{j(i)}^{-1} s(0, \ldots, 0, b, 0, \ldots, 0) \\ &= \mu_{j(i)}(a) + \mu_{j(i)}(b) \end{aligned}$$

Hence, $\mu_{j(i)} \in \Omega(P_{j(i)})$, $i = 1, \ldots, m$. Moreover, since $\lambda_i = 0$ implies $s(0, \ldots, 0, a, 0, \ldots, 0) = 0$ for every $a \in P_i$, we have

$$\begin{aligned} s(a_1, \ldots, a_n) &= s((a_1, 0, \ldots, 0) \oplus \cdots \oplus (0, \ldots, 0, a_n)) \\ &= s(a_1, 0, \ldots, 0) + \cdots + s(0, \ldots, 0, a_n) \\ &= \sum \lambda_{j(i)}\mu_{j(i)}(a_{j(i)}) \quad \blacksquare \end{aligned}$$

The following corollary shows that Theorem 3.3 cannot be strengthened to prove that a tensor chain algebra is an interval algebra.

Corollary 3.6. There exist tensor chain algebras that are not interval algebras.

Proof. Let P, Q be effect algebras where P is stateless and Q is stately. Applying Lemma 3.5, we have $\Omega(P \times Q) = \Omega(Q)$ in the sense that $s \in \Omega(P \times Q)$ if and only if $s(a, b) = \mu(b)$ for some $\mu \in \Omega(Q)$. Hence, $P \times Q$ is stately, so $P \times Q$ is a tensor chain algebra. If $P \times Q$ were an interval algebra, it would follow from Lemma 3.4 that P is an interval algebra, which is a contradiction. ∎

Corollary 3.6 shows that there exist many stately effect algebras that are not interval algebras. For example, let P be stateless and suppose $\Omega(Q)$ is order determining. Then $\Omega(P \times Q) = \Omega(Q)$, so $P \times Q$ has a large supply of states, but $P \times Q$ is not an interval algebra. Notice, however, that $\Omega(P \times Q)$ is not order determining, because $s(a, 0) = 0$ for every $a \in P$, $s \in \Omega(P \times Q)$.

Let $C = \{j2^{-n}: n, j \in \mathbb{N} \cup \{0\}, j \leq 2^n\}$. Then C is a sub-effect algebra of $[0, 1]$ and C_{2^n} is a sub-effect algebra of C for every $n \in \mathbb{N}$. The next result follows from Lemma 2.1(iv) and gives further characterizations of tensor chain algebras.

Corollary 3.7. For an effect algebra P, the following statements are equivalent. (i) P is stately, (ii) $P \otimes [0, 1]$ exists, (iii) $P \otimes C$ exists, (iv) P is a tensor chain algebra, (v) P admits a morphism into an interval algebra.

4. DOUBLING TENSOR CHAIN ALGEBRAS

An effect algebra P is a *doubling tensor chain algebra* if $P \otimes C_{2^n}$ exists for all $n \in \mathbb{N}$ and is strong on P. This section shows that a doubling tensor chain algebra is an interval algebra.

Lemma 4.1. If $P \otimes C_n$ exists, then $P \otimes C_n$ is strong on C_n.

Proof. Suppose that $1 \otimes in^{-1} \perp 1 \otimes jn^{-1}$. We now assume that $i + j > n$. Then there exists a $k \in \mathbb{N}$ with $0 < k \leq n$ such that $i + j = n + k$. Now $k \leq i$ or $k \leq j$, since otherwise $i, j < k$ and we have

$$n + k = i + j < 2k \leq n + k$$

which is a contradiction. Without loss of generality, assume that $k \leq j$. Then

$$1 \otimes \frac{i}{n} \oplus 1 \otimes \frac{j}{n} = 1 \otimes \frac{i}{n} \oplus 1 \otimes \frac{j-k}{n} \oplus 1 \otimes \frac{k}{n} = 1 \otimes 1 \oplus 1 \otimes \frac{k}{n}$$

This implies that $1 \otimes kn^{-1} = 0$ and since $1 \otimes n^{-1} \leq 1 \otimes kn^{-1}$, we have $1 \otimes n^{-1} = 0$. But then for n summands we have

$$1 \otimes 1 = 1 \otimes n^{-1} \oplus \cdots \oplus 1 \otimes n^{-1} = 0$$

which is a contradiction. Hence, $i + j \leq n$, so $in^{-1} \perp jn^{-1}$ in C_n. ∎

This last lemma shows that if P is a doubling tensor chain algebra, then $P \otimes C_{2^n}$ is strong for all $n \in \mathbb{N}$. An interval algebra $G^+[0, u]$ is *normal* if $a \in G^+$ satisfies $2^n a \leq 2^n u$ for some $n \in \mathbb{N}$, then $a \leq u$. We call $G^+[0, u]$ *regular* if $a, b \in G^+[0, u]$ satisfy $a \perp a$, $b \perp b$, then we have $a \perp b$.

Lemma 4.2. If $G^+[0, u]$ is normal, then it is regular.

Proof. Suppose $a, b \in G^+[0, u]$ satisfy $a \perp a$, $b \perp b$. Then

$$2(a + b) = 2a + 2b \leq u + u = 2u$$

Since $G^+[0, u]$ is normal, we have $a + b \leq u$. Hence, $a \perp b$, so $G^+[0, u]$ is regular. ■

Most of the common interval algebras are normal. For example, C_n, $[0, 1]$, and Hilbert space effect algebras (Foulis and Bennett, 1994; Greechie and Foulis, 1995) are normal. In fact, we shall show that if $\Omega(G^+[0, u])$ is order determining, then $G^+[0, u]$ is normal. However, the *diamond* $D = \{0, a, b, 1\}$, where $2a = 2b = 1$ and $a \not\perp b$, is an interval algebra (Bennett and Foulis, n.d.-a; Foulis *et al.*, 1994), which clearly is not regular. Hence, by Lemma 4.2, D is not normal.

Theorem 4.3. An effect algebra is a doubling tensor chain algebra if and only if it is a normal interval algebra.

Proof. Let P be a doubling tensor chain algebra. We proceed as in the proof of Theorem 3.3, where we showed that $\phi: P \to G^+[0, u]$ is a morphism, where $\phi(a) = [a]$. We now show that ϕ is an isomorphism. Suppose that $a, b \in P$ with $\phi(a) \perp \phi(b)$. Then $[a] + [b] \leq [1]$, so there exists a

$$c = c_1 \otimes 2^{-m} \oplus \cdots \oplus c_j \otimes 2^{-m} \in F$$

such that $[a + b + c] = [a] + [b] + [c] = [1]$. Hence, for some $r \in J$, we have

$$a \otimes 2^{-r} \oplus b \otimes 2^{-r} \oplus c_1 \otimes 2^{-r} \oplus \cdots \oplus c_j \otimes 2^{-r} = 1 \otimes 2^{-r}$$

Summing this equation 2^r times gives

$$a \otimes 1 \oplus b \otimes 1 \oplus c_1 \otimes 1 \oplus \cdots \oplus c_j \otimes 1 = 1 \otimes 1$$

Hence, $a \otimes 1 \perp b \otimes 1$ and since $P \otimes C_{2^r}$ is strong on P, we have $a \perp b$. Thus, ϕ is a monomorphism. To show that ϕ is surjective, suppose $0 \leq [c] \leq [1]$, where c has the previous form. Then there exists a

$$d = d_1 \otimes 2^{-n} \oplus \cdots \oplus d_k \otimes 2^{-n} \in F$$

such that $[c + d] = [c] + [d] = [1]$. Hence, for some $r \in J$, we have

$$c_1 \otimes 2^{-r} \oplus \cdots \oplus c_j \otimes 2^{-r} \oplus d_1 \otimes 2^{-r} \oplus \cdots \oplus d_k \otimes 2^{-r} = 1 \otimes 2^{-r}$$

Summing this equation 2^r times gives

$$c_1 \otimes 1 \oplus \cdots \oplus c_j \otimes 1 \oplus d_1 \otimes 1 \oplus \cdots \oplus d_k \otimes 1 = 1 \otimes 1$$

As before, we conclude that $c_1 \perp c_2$. Hence,

$$c_1 \otimes 1 \oplus \cdots \oplus c_j \otimes 1 = (c_1 \oplus c_2) \otimes 1 \oplus \cdots \oplus c_j \otimes 1$$

Continuing by induction, it follows that $c_1 \oplus \cdots \oplus c_j$ is defined. Since c is irreducible, we have

$$c = c_1 \oplus \cdots \oplus c_j \in P$$

Hence, $[c] = \phi(c)$, so ϕ is surjective and ϕ is an isomorphism. We next show that $G^+[0, u]$ is normal. Suppose $[c] \in G^+ = \hat{F}$ and $2^p[c] \leq 2^p\,[1]$ for some $p \in \mathbb{N}$. Then there exists a $d \in F$ of the previous form such that $2^p\,[c] + [d] = 2^p[1]$. Hence, for some $r \in J$, we have

$$2^p c_1 \otimes 2^{-r} \oplus \cdots \oplus 2^p c_j \otimes 2^{-r} \oplus d_1 \otimes 2^{-r} \oplus \cdots \oplus d_k \otimes 2^{-r} = 2^p 1 \otimes 2^{-r}$$

It follows that $p \leq r$ and

$$c_1 \otimes 2^p 2^{-r} \oplus \cdots \oplus c_j \otimes 2^p 2^{-r} \oplus d_1 \otimes 2^{-r} \oplus \cdots \oplus d_k \otimes 2^{-r} = 1 \otimes 2^p 2^{-r}$$

Summing this equation 2^{r-p} times gives

$$c_1 \otimes 1 \oplus \cdots \oplus c_j \otimes 1 \oplus d_1 \otimes 2^{r-p} 2^{-r} \oplus \cdots \oplus d_k \otimes 2^{r-p} 2^{-r} = 1 \otimes 1$$

As before, we conclude that $c \in P$, so $[c] \leq [1]$ and $G^+[0, u]$ is normal.

Conversely, let $G^+[0, u]$ be a normal interval algebra and let ψ: $P \to G^+[0, u]$ be an isomorphism. For $n \in \mathbb{N}$, it is clear that $G^+[0, 2^n u]$ is an interval algebra. Define the bimorphism β: $P \times C_{2^n} \to G^+[0, 2^n u]$ by $\beta(a, j2^{-n}) = j\psi(a)$. Suppose that $\beta(a, 1) \perp \beta(b, 1)$. Then $\beta(a, 1) \oplus \beta(b, 1)$ is defined so $2^n\psi(a) \oplus 2^n\psi(b)$ is defined in $G^+[0, 2^n u]$. Hence, $\psi(a) \oplus \psi(b)$ is defined in $G^+[0, 2^n u]$ and we have

$$2^n(\psi(a) \oplus \psi(b)) = 2^n\psi(a) \oplus 2^n\psi(b) \leq 2^n u$$

Since $G^+[0, u]$ is normal, we have $\psi(a) \oplus \psi(b) \leq u$. It follows that $\psi(a) \perp \psi(b)$ in $G^+[0, u]$. Since ψ is an isomorphism, we conclude that $a \perp b$, so β is strong on P. Applying Lemma 2.1(i), it follows that $P \otimes C_{2^n}$ is strong on P. Hence, P is a doubling tensor chain algebra. ■

The following result strengthens a theorem in Bennett and Foulis (n.d.-a).

Corollary 4.4. If $\Omega(P)$ is order determining, then P is a normal interval algebra.

Proof. This follows from Lemma 2.1(ii) and Theorem 4.3. ■

The converse of Corollary 4.4 does not hold. For example, the nonstandard unit interval *[0, 1] is a normal interval algebra. However, Ω(*[0, 1]) contains only one element and this state vanishes on the infinitesimals.

5. DOUBLINGS

Let $(P, \oplus, 0, 1)$ be an effect algebra and suppose there exists a $u \in P$ such that $2^n u = 1$ for some $n \in \mathbb{N}$. If an effect algebra Q admits a morphism into $P[0, u]$, we call P a *weak n-doubling* of Q and if Q admits an isomorphism onto $P[0, u]$, we call P an *n-doubling* of Q. Moreover, we call u an *n-doubling unit* if $2^n a$ exists implies $a \leq u$. Finally, if Q admits an isomorphism onto $P[0, u]$, where u is an n-doubling unit, we call P a *strong n-doubling* of Q. If $n = 1$, we refer to these as (weak, strong) doublings. Doublings have been previously considered in Bennett and Foulis (n.d.-b).

If $Q = G^+[0, u]$ is an interval algebra, then $G^+[0, 2^n u]$ is an n-doubling of Q, so every interval algebra admits an n-doubling for every $n \in \mathbb{N}$. However, the next result shows that an interval algebra need not admit a strong n-doubling.

Theorem 5.1. For an effect algebra Q, the following statements are equivalent. (i) Q is a normal interval algebra. (ii) Q admits a strong n-doubling for every $n \in \mathbb{N}$. (iii) Q is a doubling tensor chain algebra.

Proof. To show that (i) implies (ii), let $G^+[0, u]$ be a normal interval algebra. If $a, 2^n a \in G^+[0, 2^n u]$, then $2^n a \leq 2^n u$. Since $G^+[0, u]$ is normal, $a \leq u$. Hence, u is an n-doubling unit, so $G^+[0, 2^n u]$ is a strong n-doubling of $G^+[0, u]$. If Q is isomorphic to $G^+[0, u]$, we conclude that Q admits a strong n-doubling for every $n \in \mathbb{N}$. To show that (ii) implies (iii), suppose $\phi: Q \to P[0, u]$ is an isomorphism, where u is an n-doubling unit. First notice that if $c \in P[0, u]$ and $j \in \mathbb{N}$ with $j \leq 2^n$, then jc is defined in P. Indeed, since $c \oplus c' = u$, we have

$$2^n c \oplus 2^n c' = 2^n(c \oplus c') = 2^n u = 1$$

Hence, $2^n c$ is defined, so jc is also defined in P. It follows that the map β: $Q \times C_{2^n} \to P$ given by $\beta(a, j2^{-n}) = j\phi(a)$ is well defined. Since β is a bimorphism, $Q \otimes C^{2^n}$ exists. Suppose that $\beta(a, 1) \perp \beta(b, 1)$. Then $\beta(a, 1) \oplus \beta(b, 1)$ is defined, so

$$2^n(\phi(a) \oplus \phi(b)) = 2^n\phi(a) \oplus 2^n\phi(b)$$

is defined in P. Since u is an n-doubling unit, we have $\phi(a) \oplus \phi(b) \leq u$, so $\phi(a) \perp \phi(b)$ in $P[0, u]$. Since ϕ is an isomorphism, $a \perp b$, so β is strong on Q. Applying Lemma 2.1(i), we conclude that $P \otimes C_{2^n}$ is strong on P. Hence, Q is a doubling tensor chain algebra. That (iii) implies (i) follows from Theorem 4.3. ∎

We now study the relationship between chain tensor products and weak n-doublings.

Theorem 5.2. If $P \otimes C_{2^n}$ exists, then $P \otimes C_{2^n}$ is a weak n-doubling of P. Conversely, if a weak n-doubling of P exists, then $P \otimes C_{2^n}$ exists.

Proof. Suppose that $P \otimes C_{2^n}$ exists and let $u = 1 \otimes 2^{-n} \in P \otimes C_{2^n}$. Then

$$2^n u = 1 \otimes 1 = 1 \in P \otimes C_{2^n}$$

Let $Q = P \otimes C_{2^n}[0, u]$ and define $\phi: P \to Q$ by $\phi(a) = a \otimes 2^{-n}$. Notice that $a \otimes 2^{-n} \leq u$, so indeed $\phi(a) \in Q$. Moreover, $P \otimes C_{2^n}$ is an n-doubling of Q. To show that ϕ is a morphism, we have $\phi(1) = 1 \otimes 2^{-n} = u$. Now suppose that $a, b \in P$ with $a \perp b$. Then $a \otimes 2^{-n} \perp b \otimes 2^{-n}$, so $\phi(a) \perp \phi(b)$ in $P \otimes C_{2^n}$. Also,

$$\phi(a) \oplus \phi(b) = a \otimes 2^{-n} \oplus b \otimes 2^{-n} = (a \oplus b) \otimes 2^{-n} \leq 1 \otimes 2^{-n} = u$$

Hence, $\phi(a) \perp \phi(b)$ in Q and

$$\phi(a \oplus b) = (a \oplus b) \otimes 2^{-n} = \phi(a) \oplus \phi(b)$$

Therefore, ϕ is a morphism, so $P \otimes C_{2^n}$ is a weak n-doubling of P.

Conversely, suppose Q is a weak n-doubling of P. Then there exists a morphism $\phi: P \to Q[0, u]$ where $nu = 1$. Define $\beta: P \times C_{2^n} \to Q$ by $\beta(a, j2^{-n}) = j\phi(a)$. Proceeding as in the proof of Theorem 5.1, we conclude that β is a bimorphism. Hence, $P \otimes C_{2^n}$ exists. ∎

Corollary 5.3. For an effect algebra P, the following statements are equivalent. (i) P is a tensor chain algebra, (ii) P admits a weak n-doubling for all $n \in$ N, (iii) P admits a morphism into an interval algebra.

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